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OPTIMAL INSTRUMENTATION OF UNCERTAIN STRUCTURAL SYSTEMS SUBJECT TO EARTHQUAKE GROUND MOTIONS

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SUMMARY

A criterion is proposed for making decisions regarding the optimal location of a given number of sensors to record the seismic response of a structure for identification purposes. The optimal location of the sensors is selected so that the expected value of a Bayesian loss function, expressed in terms of the Fisher information in the recordings, is minimized. The criterion is applied to the case of multi-degree-of-freedom systems with uncertain structural properties subjected to earthquake ground motions modelled as stationary stochastic processes. The use and capabilities of the criterion are thoroughly illustrated by means of an example. Results are used to assess the influence of record duration, recording noise, and ground motion frequency content and amplitude, on the optimal location of accelerometers as well as on the reduction of prior uncertainty about the structural parameters. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: sensor location; optimal instrumentation; Fisher information; system identification; random vibration; seismic response

INTRODUCTION

Our interest in the identification of the mechanical properties of civil engineering structures subjected to seismic ground motion of different intensities has several sources. The information provided by records of the dynamic response of full-size structures under the action of real earthquakes is of great value in the development of our capability to predict the expected response of new structures while they are in the design stage; and a collection of records obtained on a structure during a sequence of earthquakes can help us to evaluate the influence of damage accumulation on the mechanical properties of that structure.

Due to the limitations of the mathematical models used to represent the behaviour of real, complex systems, and because of the limited number of records that can be obtained simultaneously during a seismic event, which in usual cases is much smaller than the number of degrees of freedom of the structure of interest, the mechanical properties estimated from the analysis of the recorded responses are usually tied to significant uncertainties. Hence, our interest in the study of the concepts that affect those uncertainties, as well as in the development of optimum instrumentation schemes, i.e. selection of the locations of the recording instruments on a structure that lead to the smallest probabilistic deviations of the values estimated for the system's properties from the real ones.

Some research has been carried out to develop methodologies for selecting the optimal location of sensors for identification of dynamic systems (see e.g. Reference 1). Criteria for locating recording instruments to

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assure global convergence and uniqueness of the inverse problem associated with parameter identification have also been established.[2,3] More recently, a methodology has been formulated by Udwadia[4] to solve the optimum sensor location problem. It is advantageous in that the optimization is uncoupled from the identification problem through the concept of efficient estimation. The criterion presented here also makes use of efficient estimators but explicitly takes into account the uncertainty about the structural parameters and the seismic ground motion excitation; in this sense it could be thought of as an extension of the methodology developed by Udwadia. Current knowledge on the structural parameters to be identified from recordings is considered to be summarized by prior probability density functions and the ground motion at the base of the structural systems is described in terms of spectral density functions. The paper first proposes a criterion for selecting optimal locations of a given number of recording instruments based on minimizing an expected loss function. Through the use of efficient estimation theory, the loss function is related to the Fisher information associated with the responses to be recorded. Expressions for applying such criterion are then rigorously derived for the case of linear stochastic structural response. The paper then goes on to show a detailed example for illustration of the use and features of the formulation proposed. Results are thoroughly discussed and conclusions and recommendations are given at the end.

OPTIMIZING CRITERION

Consider the case where a vector $\Theta = \{\theta_1, \theta_2, \ldots, \theta_K\}$ of K uncertain parameters θ_i, $i = 1, 2, \ldots, K$, is to be estimated based on the observation of a set of random vectors $Y_0 = \{\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_N\}$. Suppose that given certain constraints on the number of vectors that can be observed, a subset Y of M vectors out of the N vectors in Y_0 has to be chosen for the estimation of Θ. It is desirable to select for observation the M vectors that would yield the 'best' estimate of the parameter vector Θ.

Let $f(y\,|\,\theta)$ denote the conditional joint probability density function of the random vectors Y. An unbiased estimator $\hat{\Theta}(Y)$ of Θ, $E_{Y|\Theta}[\hat{\Theta}(Y)] = \Theta$, is said to be efficient if its conditional covariance matrix $\mathrm{Cov}(\hat{\Theta}(Y)\,|\,\Theta)$ is given by

$$\mathrm{Cov}(\hat{\Theta}(Y)\,|\,\Theta) = M_\Theta^{-1} \tag{1}$$

In equation (1), M_Θ is the Fisher information matrix defined as

$$M_\Theta = E_{Y|\Theta}\left[\left\{\frac{\partial}{\partial\Theta}\ln f(Y\,|\,\Theta)\right\}\left\{\frac{\partial}{\partial\Theta}\ln f(Y\,|\,\Theta)\right\}^{\mathrm{T}}\right] \tag{2}$$

where $\partial \ln f(Y\,|\,\Theta)/\partial\Theta$ is the vector

$$\frac{\partial}{\partial\Theta}\ln f(Y\,|\,\Theta) = \left\{\frac{\partial}{\partial\theta_i}\ln f(Y\,|\,\Theta)\right\}, \quad i = 1, 2, \ldots, K \tag{3}$$

and the superscript 'T' denotes vector transpose. The inverse of the Fisher information matrix (FIM) is the Cramer–Rao lower bound and represents the minimum covariance that an unbiased estimator can achieve. Equation (1) suggests that the greater the size of the FIM, as measured by suitable norms of the matrix, the smaller the size of the estimator covariance matrix. The size of the FIM gets larger as the random function $f(Y\,|\,\Theta)$ becomes more sensitive to changes in the values of the parameters in Θ.

Suppose now that we can assign to Θ a prior distribution which summarizes the information and knowledge that one has on the likelihood of the possible values of Θ before Y is observed. Each subset Y of M vectors is a collection of observations from a possible 'instrumentation' for the estimation of Θ. In order to compare between different instrumentation alternatives, a measure of the goodness of each alternative—related to the expected accuracy of the parameter estimates to be obtained from the observed Y—is

required. A suitable measure of goodness can be defined in terms of a Bayesian loss function $L(\Theta, \hat{\Theta}(Y))$ for which the expectation $E[L(\Theta, \hat{\Theta}(Y))]$ is to be minimized. Based on a Taylor series expansion of $L(\Theta, \hat{\Theta}(Y))$ about Θ it may be shown that[4]

$$E[L(\Theta, \hat{\Theta}(Y))] \approx \frac{1}{2} E_{\Theta}\left[\operatorname{tr} \frac{\partial^2 L}{\partial \hat{\Theta}^2} \operatorname{Cov}(\hat{\Theta}(Y) \mid \Theta) \right] \tag{4}$$

and since $\hat{\Theta}(Y)$ is an efficient estimator, it follows from equation (1) that

$$E[L(\Theta, \hat{\Theta}(Y))] \approx \frac{1}{2} E_{\Theta}\left[\operatorname{tr} \frac{\partial^2 L}{\partial \hat{\Theta}^2} M_{\Theta}^{-1} \right] \tag{5}$$

The expectations on the left-hand side of equations (4) and (5) are taken over Y and Θ jointly. A commonly used loss function is the so-called squared error loss function,

$$L(\Theta, \hat{\Theta}(Y)) = (\hat{\Theta} - \Theta)^{\mathrm{T}} (\hat{\Theta} - \Theta) \tag{6}$$

For the squared error loss function, the second derivative of L with respect to $\hat{\Theta}$ is equal to two times the identity matrix and equation (5) becomes an equality; thus,

$$E[(\hat{\Theta} - \Theta)^{\mathrm{T}} (\hat{\Theta} - \Theta)] = E_{\Theta}[\operatorname{tr} M_{\Theta}^{-1}] \tag{7}$$

The criterion to select the optimal instrumentation alternative is then to choose for observation the subset Y for which the expected loss in equation (7) is a minimum. The expected value on the right-hand side of equation (7) is taken with respect to Θ, i.e. with respect to the prior distribution of Θ. Therefore, selecting the optimal alternative involves the prior knowledge that one has about the properties of the system. Notice that whereas the criterion proposed by Udwadia[4] maximizes the size of the Fisher information matrix, the one proposed here is based on minimizing the expected value of the Bayesian loss function.

OPTIMAL INSTRUMENTATION OF MDOF STRUCTURAL SYSTEMS

Let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_N$ be M-dimensional, independent, Gaussian, zero-mean, random vectors. Let θ_k be any uncertain parameter in Θ; then the partial derivative of the logarithm of the conditional joint density function $f(Y \mid \Theta) = f(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_N \mid \Theta)$ can be written as

$$\frac{\partial \ln f(Y \mid \Theta)}{\partial \theta_k} = -\frac{1}{2} \sum_{i=1}^{N} \left\{ \frac{\partial \ln \Delta_i}{\partial \theta_k} + \mathbf{Y}_i^{\mathrm{T}} \frac{\partial \mathbf{C}_i^{-1}}{\partial \theta_k} \mathbf{Y}_i \right\} \tag{8}$$

where $\mathbf{C}_i = E_{Y \mid \Theta}[\mathbf{Y}_i \mathbf{Y}_i^{\mathrm{T}}]$ is a covariance matrix and Δ_i denotes its determinant. If M_{kl} denotes an element in the kth row and lth column of the Fisher information matrix, then by definition

$$\begin{aligned} M_{kl} &= E_{Y \mid \Theta}\left[\frac{\partial \ln f(Y \mid \Theta)}{\partial \theta_k} \frac{\partial \ln f(Y \mid \Theta)}{\partial \theta_l} \right] \\ &= \frac{1}{4} E_{Y \mid \Theta}\left[\sum_{i=1}^{N} \sum_{j=1}^{N} \left\{ \frac{\partial \ln \Delta_i}{\partial \theta_k} + \mathbf{Y}_i^{\mathrm{T}} \frac{\partial \mathbf{C}_i^{-1}}{\partial \theta_k} \mathbf{Y}_i \right\} \left\{ \frac{\partial \ln \Delta_j}{\partial \theta_l} + \mathbf{Y}_j^{\mathrm{T}} \frac{\partial \mathbf{C}_j^{-1}}{\partial \theta_l} \mathbf{Y}_j \right\} \right] \end{aligned} \tag{9}$$

It can be shown (see Appendix I) that

$$E_{Y \mid \Theta}\left[\frac{\partial \ln \Delta_i}{\partial \theta_k} + \mathbf{Y}_i^{\mathrm{T}} \frac{\partial \mathbf{C}_i^{-1}}{\partial \theta_k} \mathbf{Y}_i \right] = 0 \tag{10}$$

Thus, using equation (10) and given that $\mathbf{Y}_i$ and $\mathbf{Y}_j$ are statistically independent for $i \neq j$, equation (9) can be written as follows:

$$M_{kl} = \frac{1}{4} E_{Y|\Theta}\left[\sum_{i=1}^{N}\left\{\frac{\partial \ln \Delta_i}{\partial \theta_k} + \mathbf{Y}_i^{\mathrm{T}} \frac{\partial \mathbf{C}_i^{-1}}{\partial \theta_k} \mathbf{Y}_i\right\}\left\{\frac{\partial \ln \Delta_i}{\partial \theta_l} + \mathbf{Y}_i^{\mathrm{T}} \frac{\partial \mathbf{C}_i^{-1}}{\partial \theta_l} \mathbf{Y}_i\right\}\right] \tag{11}$$

The expected values in equation (11) involve fourth-order moments which can be obtained from the second order ones provided that $\mathbf{Y}_i$ is a Gaussian vector. In Appendix II it is shown that

$$E_{Y|\Theta}\left[\mathbf{Y}_i^{\mathrm{T}} \frac{\partial \mathbf{C}_i^{-1}}{\partial \theta_k} \mathbf{Y}_i \mathbf{Y}_i^{\mathrm{T}} \frac{\partial \mathbf{C}_i^{-1}}{\partial \theta_l} \mathbf{Y}_i\right] = 2\mathbf{C}_i^{-1} \frac{\partial \mathbf{C}_i}{\partial \theta_k} \cdot \frac{\partial \mathbf{C}_i}{\partial \theta_l} \mathbf{C}_i^{-1} + \frac{\partial \ln \Delta_i}{\partial \theta_k} \frac{\partial \ln \Delta_i}{\partial \theta_l} \tag{12}$$

where the dot in equation (12) denotes a scalar matrix product. Substituting equation (12) into equation (11) and making use of equation (10), one obtains for M_{kl}:

$$M_{kl} = \frac{1}{2} \sum_{i=1}^{N} \mathbf{C}_i^{-1} \frac{\partial \mathbf{C}_i}{\partial \theta_k} \cdot \frac{\partial \mathbf{C}_i}{\partial \theta_l} \mathbf{C}_i^{-1} \tag{13}$$

Consider now a linear structural system with Q degrees of freedom subjected to an earthquake ground motion modelled as a zero-mean stationary Gaussian stochastic process. Suppose there are $M < Q$ recording instruments available to record the system response. Let the process $Y_i(t_n)$, $i = 1, 2, \ldots, M$, denote the ith recorded response at discrete times $t_n = (n-1)\Delta t$, $n = 1, 2, \ldots, N$. Under the assumption of mean-square continuity, $Y_i(t_n)$ can be expressed as a sum of independent frequency-specific processes in consecutive constant-size frequency intervals as follows[5]:

$$Y_i(t_n) = \sum_{k=1}^{N} A_{ik} \cos(\omega_k t_n) + B_{ik} \sin(\omega_k t_n) \tag{14}$$

where the Fourier coefficients A_{ik}, B_{ik}, corresponding to frequency $\omega_k = (k-1)2\pi/N\Delta t$, are random variables. The coefficients A_{ik}, B_{ik}, are related to $Y_i(t_n)$ through the discrete Fourier transform:

$$\begin{aligned} A_{ik} &= \frac{1}{N} \sum_{n=1}^{N} Y_i(t_n) \cos\left(\frac{2\pi(k-1)(n-1)}{N}\right) \\ B_{ik} &= \frac{1}{N} \sum_{n=1}^{N} Y_i(t_n) \sin\left(\frac{2\pi(k-1)(n-1)}{N}\right) \end{aligned} \tag{15}$$

Notice that the recorded response $Y_i(t_n)$ is a periodic stationary stochastic process with physical length $L = (N-1)\Delta t$ and period $T = N\Delta t$. In this modelling, we could think of L as the record duration.

Let A_{jk}, B_{jk} denote the Fourier coefficients of the jth recorded response $Y_j(t_n)$. The covariance between coefficients A_{ik}, B_{ik} and A_{jk}, B_{jk} can be obtained using the expressions in equation (15):[6,7]

$$E[A_{ik} A_{jk}] = \begin{cases} \frac{1}{2} R_{ij}(\omega_k)\Delta\omega & \text{for } k = 1 \\ \frac{1}{4}\{R_{ij}(\omega_k) + R_{ij}(\omega_{N-k+2})\}\Delta\omega & \text{for } k = 2, \ldots, N/2 \\ R_{ij}(\omega_k)\Delta\omega & \text{for } k = 1 + N/2 \end{cases} \tag{16}$$

$$E[A_{ik} B_{jk}] = \begin{cases} \frac{1}{4}\{Q_{ij}(\omega_k) - Q_{ij}(\omega_{N-k+2})\}\Delta\omega & \text{for } k = 2, \ldots, N/2 \\ 0 & \text{for } k = 1, 1 + N/2 \end{cases} \tag{17}$$

and $E[B_{ik}\ B_{jk}] = E[A_{ik}\ A_{jk}]$ for $k = 2, 3, \ldots, N/2$, $E[B_{ik}\ B_{jk}] = 0$ for $k = 1,\ 1 + N/2$; when $k = 1 + N/2$, ω_k is the Nyquist frequency. In equations (16) and (17), $R_{ij}(\omega_k)$ and $Q_{ij}(\omega_k)$ are the real and imaginary parts of the cross-spectral density function of recorded responses Y_i and Y_j, $SY_{ij}(\omega_k)$, respectively,

$$SY_{ij}(\omega_k) = R_{ij}(\omega_k) + \imath Q_{ij}(\omega_k), \quad \imath = \sqrt{-1} \tag{18}$$

The recorded response $Y_i(t_n)$ can be expressed in terms of the system response $X_i(t_n)$ and a recording noise $N_i(t_n)$ as follows:

$$Y_i(t_n) = X_i(t_n) + N_i(t_n) \tag{19}$$

where the system response is a zero-mean stationary Gaussian process and the recording noises are modelled as independent, zero-mean, Gaussian, stationary, white noise processes with spectral density amplitude N_0.

From equation (19) it follows that $SY_{ij}(\omega_k)$ can be written as

$$SY_{ij}(\omega_k) = S_{ij}(\omega_k) + N_0\delta_{ij} \tag{20}$$

where $S_{ij}(\omega_k)$ is the complex cross-spectral density function of system responses $X_i(t_n)$ and $X_j(t_n)$, N_0 is the recording noise spectral density function and δ_{ij} is the Kronecker delta. Notice that $SY_{ij}(\omega_k) = S_{ij}(\omega_k)$ for $i \neq j$; the presence of noise in the recordings affects the spectral density functions only, thus

$$SY_{ii}(\omega_k) = S_{ii}(\omega_k) + N_0 \tag{21}$$

Let $\mathbf{F}_k^{\mathrm{T}} = \{\mathbf{A}_k, \mathbf{B}_k\}^{\mathrm{T}}$ denote the $(1 \times 2M)$ vector of recorded response Fourier coefficients at the M recording points corresponding to frequency ω_k, where $\mathbf{A}_k = \{A_{1k}, A_{2k}, \ldots, A_{Mk}\}$, $\mathbf{B}_k = \{B_{1k}, B_{2k}, \ldots, B_{Mk}\}$. The Fourier coefficients $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N$ are independent, zero-mean, Gaussian random vectors. Using the expressions in equations (16) and (17), the $(2M \times 2M)$ covariance matrix $\mathbf{C}_k = E[\mathbf{F}_k\mathbf{F}_k^{\mathrm{T}}]$ can be assembled as follows:

$$\mathbf{C}_k = \begin{bmatrix} \mathbf{C}_{AA} & \mathbf{C}_{AB} \\ \mathbf{C}_{AB}^{\mathrm{T}} & \mathbf{C}_{BB} \end{bmatrix} \tag{22}$$

where matrices $\mathbf{C}_{AA} = E[\mathbf{A}_k\mathbf{A}_k^{\mathrm{T}}]$, $\mathbf{C}_{AB} = E[\mathbf{A}_k\mathbf{B}_k^{\mathrm{T}}]$, $\mathbf{C}_{BB} = E[\mathbf{B}_k\mathbf{B}_k^{\mathrm{T}}]$ for each frequency ω_k, $k = 1, 2, \ldots, N$.

Once the covariance matrices $\mathbf{C}_k$, $k = 1, 2, \ldots, N$, for the Fourier coefficients are obtained, equation (13) for a set of independent Gaussian random vectors can be used to evaluate the elements of the Fisher information matrix,

$$M_{pq} = \frac{1}{2}\sum_{k=1}^{N} \mathbf{C}_k^{-1}\frac{\partial \mathbf{C}_k}{\partial \theta_p} \cdot \frac{\partial \mathbf{C}_k}{\partial \theta_q}\mathbf{C}_k^{-1} \tag{23}$$

where θ_p and θ_q are two of the uncertain structural parameters, N is the number of discrete points in the records, $\mathbf{C}_k$ are covariance matrices of order $(2M \times 2M)$, and M is the number of available recording instruments. Equation (23) shows that the Fisher information matrix depends on the degrees of correlation between recorded responses, as measured by the covariance matrices, and on the sensitivity of such degrees of correlation to the uncertain structural parameters, as accounted for by the partial derivatives of the covariance matrices. The FIM can be thought of as a weighted combination of the sensitivities of the degrees of correlation between recorded responses, with the weights being the inverse covariance matrices of such responses. Conceptually, since the objective is to maximize the size of the FIM, more weight is given in equation (23) to those responses that are more correlated and whose covariances are more sensitive to small changes in the parameters to be estimated.

In order to select the optimal instrumentation alternative one then proceeds as follows. First, for each instrumentation alternative of the MDOF structural system, i.e. for each possible combination of recorded responses Y_i, $i = 1, 2, \ldots, M$, the corresponding Fisher information matrix is assembled based on equation (23). As stated earlier, the elements in the FIM will in general depend on the uncertain parameters in Θ. The trace of the FIM inverse is obtained next and its expected value with respect to the prior distribution of Θ is computed. According to the criterion proposed here, the optimal instrumentation alternative will then be the set of M recorded responses Y_i for which such expected value is a minimum.

STOCHASTIC STRUCTURAL RESPONSE

Next, we turn to the evaluation of the cross-spectral density function of a pair of system responses, $S_{ij}(\omega)$. Consider two response components X_i and X_j of a linear structural system with Q degrees of freedom, mass and stiffness matrices $\mathbf{M}$ and $\mathbf{K}$, modal frequencies and critical damping ratios ω_q, ξ_q, and mode shapes ϕ_q, $q=1,2,\ldots,Q$. The cross-spectral density function of the response components X_i and X_j, $S_{ij}(\omega)$, is given by[8]

$$S_{ij}(\omega) = S_{UU}(\omega) + \sum_{q=1}^{Q} c_q^j H_q^*(\omega) S_{U\ddot{U}}(\omega) + \sum_{q=1}^{Q} c_q^i H_q(\omega) S_{\ddot{U}U}(\omega) + \sum_{q=1}^{Q} \alpha_q^{ij} |H_q(\omega)|^2 S_{\ddot{U}\ddot{U}}(\omega) \tag{24}$$

where S_{UU} and $S_{\ddot{U}\ddot{U}}$ are the ground displacement and acceleration spectral density functions, respectively, $S_{U\ddot{U}}(\omega)$ is the cross-spectrum between ground displacement U and ground acceleration $\ddot{U}$, $H_q(\omega)$ is the modal transfer function

$$H_q(\omega) = \frac{1}{\omega_q^2 - \omega^2 + 2\imath\xi_q\omega_q\omega}, \quad \imath = \sqrt{-1} \tag{25}$$

and $H_q^*(\omega)$ is the complex conjugate. The first term on the right-hand side of equation (24) corresponds to the contribution from the pseudo-static response, the fourth term corresponds to that from the dynamic response, whereas the second and third terms represent the contribution from the cross-correlation between them. In equation (24), c_q^i is an effective modal participation factor associated with response X_i,

$$c_q^i = -\mathbf{r}_i^{\mathrm{T}} \phi_q \frac{\phi_q^{\mathrm{T}}\mathbf{MJ}}{\phi_q^{\mathrm{T}}\mathbf{M}\phi_q} \tag{26}$$

and $\mathbf{r}_i$ denotes a response transfer vector which relates the response component X_i with the system response displacements $\mathbf{x}$ along degrees of freedom, $\mathbf{x} = \{x_1, x_2, \ldots, x_Q\}$, $X_i = \mathbf{r}_i^{\mathrm{T}}\mathbf{x}$; $\mathbf{J}$ is a column vector of ones. The coefficients α_q^{ij} in equation (24) are equal to[6]

$$\alpha_q^{ij} = c_q^i c_q^j + \frac{1}{2} \sum_{k=1, k\neq q}^{Q} (c_q^i c_k^j + c_k^i c_q^j) \mathscr{A}_{qk} \tag{27}$$

The factor $\mathscr{A}_{qk}$ in equation (27) depends only on the modal frequencies and dampings associated with the qth and kth modes, and takes into account the contribution to the system response from the cross-correlation between them. For a system with well-spaced modal frequencies or light modal dampings, $\mathscr{A}_{qk}$ is approximately zero and can be neglected in equation (27).

If response components X_i and X_j are total displacements along the response degrees of freedom of the structural system, the effective modal participation factors in equation (26) are equal to

$$\begin{aligned} c_q^i &= -\phi_{iq} \frac{\phi_q^{\mathrm{T}}\mathbf{MJ}}{\phi_q^{\mathrm{T}}\mathbf{M}\phi_q} \\ c_q^j &= -\phi_{jq} \frac{\phi_q^{\mathrm{T}}\mathbf{MJ}}{\phi_q^{\mathrm{T}}\mathbf{M}\phi_q} \end{aligned} \tag{28}$$

where ϕ_{iq} denotes the ith component of the modal shape vector ϕ_q.

It is usually the case that the recorded response components are total accelerations along the response degrees of freedom. In such a case, the cross-spectral density function of total accelerations can be obtained

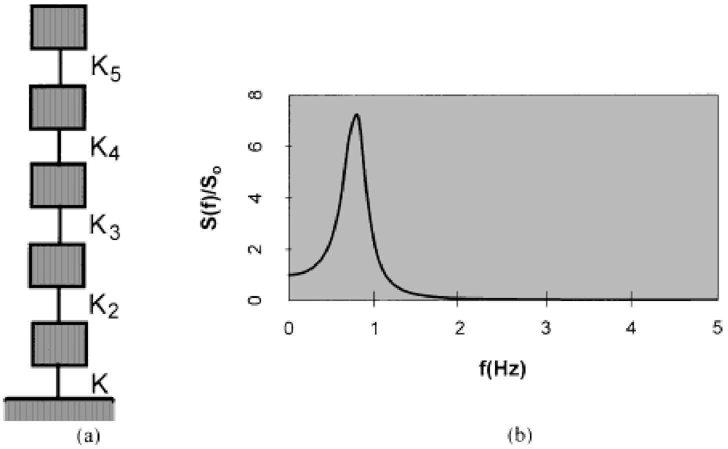

Figure 1. (a) Five degree-of-freedom structural system; (b) earthquake ground motion spectral density function, $\omega_g = 0{\cdot}8$ Hz, $\xi_g = 0{\cdot}20$

from equation (24) using the relations for the cross-spectrum of a derivative process,

$$S_{ij}(\omega) = \omega^4 \left[S_{UU}(\omega) + \sum_{q=1}^{Q} c_q^j H_q^*(\omega) S_{U\ddot{U}}(\omega) + \sum_{q=1}^{Q} c_q^i H_q(\omega) S_{\ddot{U}U}(\omega) + \sum_{q=1}^{Q} \alpha_q^{ij} |H_q(\omega)|^2 S_{\ddot{U}\ddot{U}}(\omega) \right] \qquad (29)$$

Notice though that in equation (29), the factors c_q^i, c_q^j are as given in equation (28) for displacement response components. Taking into account that $S_{\ddot{U}\ddot{U}}(\omega) = \omega^4 S_{UU}(\omega)$, $S_{\ddot{U}\ddot{U}}(\omega) = -\omega^2 S_{U\ddot{U}}(\omega)$, equation (29) can be rewritten as

$$S_{ij}(\omega) = S_{\ddot{U}\ddot{U}}(\omega) \left[1 - \sum_{q=1}^{Q} c_q^j \omega^2 H_q^*(\omega) - \sum_{q=1}^{Q} c_q^i \omega^2 H_q(\omega) + \sum_{q=1}^{Q} \alpha_q^{ij} |\omega^2 H_q(\omega)|^2 \right] \qquad (30)$$

Expression (30) is used in equation (20) for computing the covariances between Fourier coefficients given by equations (16) and (17). It is also used to obtain the covariance matrix derivatives in equation (23) for the evaluation of the Fisher information matrix.

EXAMPLES

Consider the five degree-of-freedom shear building shown in Figure 1(a) having uncertain lateral inter-storey stiffnesses. Masses at the floor levels are taken equal to 1 K sec^2/in and a critical damping ratio of 2 per cent is considered for all modes. The system is subjected to a Gaussian ground acceleration with the Kanai-Tajimi spectral density function shown in Figure 1(b); the characteristic soil frequency, ω_g, and critical damping ratio, ξ_g, for the Kanai-Tajimi model are equal to 0·8 Hz and 0·20, respectively. Let K denote the uncertain lateral stiffness of the first inter-storey and K_i, $i = 2, \ldots, 5$ denote the stiffnesses of the upper inter-storeys. For the purpose of illustration, results will be given for the case of perfectly correlated lateral stiffnesses, i.e. $K_i = c_i K$ where c_i, $i = 2, \ldots, 5$ are constants. It should be noticed, however, that the formulation developed above is of a general nature and is not restricted to the case of perfectly correlated structural parameters.

In order to compute the Fisher information according to equation (23), it is necessary to evaluate the partial derivatives of the covariance matrices $\mathbf{C}_k$ given in equation (22). The elements of these covariance matrices depend on the cross-spectral density functions of system responses, $S_{ij}(\omega)$, as shown in equations (16)–(20). Suppose that the recording instruments available are accelerometers that will record total

accelerations along the response degrees of freedom of the structure. One needs to evaluate the partial derivative of $S_{ij}(\omega)$ in equation (30) with respect to the uncertain first-storey lateral stiffness K:

$$\frac{\partial S_{ij}(\omega)}{\partial K} = S_{\ddot{U}\ddot{U}}(\omega)\left[-\sum_{q=1}^{Q}\frac{\partial c_q^j}{\partial K}\omega^2 H_q^*(\omega) - \sum_{q=1}^{Q} c_q^j \omega^2 \frac{\partial H_q^*(\omega)}{\partial K} - \sum_{q=1}^{Q}\frac{\partial c_q^i}{\partial K}\omega^2 H_q(\omega)\right.$$
$$\left. - \sum_{q=1}^{Q} c_q^i \omega^2 \frac{\partial H_q(\omega)}{\partial K} + \sum_{q=1}^{Q}\frac{\partial \alpha_q^{ij}}{\partial K}|\omega^2 H_q(\omega)|^2 + \sum_{q=1}^{Q}\alpha_q^{ij}\frac{\partial |\omega^2 H_q(\omega)|^2}{\partial K}\right] \tag{31}$$

The partial derivatives on the right-hand side of equation (31) are obtained as follows:
(a) From equation (25) for the modal transfer functions $H_q(\omega)$,

$$\frac{\partial H_q(\omega)}{\partial K} = |H_q(\omega)|^2\left\{\frac{\partial \omega_q^2}{\partial K} - 2\imath\xi_q\omega\frac{\partial \omega_q}{\partial K}\right\} - \{(\omega_q^2-\omega^2) - 2\imath\xi_q\omega_q\omega\}2(\omega_q^2-\omega^2) + 4\xi_q^2\omega^2|H_q(\omega)|^4\frac{\partial \omega_q^2}{\partial K} \tag{32}$$

$$\frac{\partial |\omega^2 H_q(\omega)|^2}{\partial K} = -\omega^4\frac{2(\omega_q^2-\omega^2)+4\xi_q^2\omega^2}{[(\omega_q^2-\omega^2)^2+4\xi_q^2\omega_q^2\omega^2]^2}\frac{\partial \omega_q^2}{\partial K} \tag{33}$$

(b) From equation (27) for the modal coefficients α_q^{ij},

$$\frac{\partial \alpha_q^{ij}}{\partial K} = \frac{\partial c_q^i}{\partial K}c_q^j + c_q^i\frac{\partial c_q^j}{\partial K} \tag{34}$$

(c) From equation (28) for the effective modal participation factors c_q^i,

$$\frac{\partial c_q^i}{\partial K} = -\frac{\partial \phi_{iq}}{\partial K}\frac{\phi_q^{\mathrm{T}}\mathbf{MJ}}{\phi_q^{\mathrm{T}}\mathbf{M}\phi_q} - \phi_{iq}\frac{\partial}{\partial K}\left\{\frac{\phi_q^{\mathrm{T}}\mathbf{MJ}}{\phi_q^{\mathrm{T}}\mathbf{M}\phi_q}\right\}$$
$$= -\frac{\partial \phi_{iq}}{\partial K}\frac{\phi_q^{\mathrm{T}}\mathbf{MJ}}{\phi_q^{\mathrm{T}}\mathbf{M}\phi_q} - \phi_{iq}\left\{\frac{\partial \phi_q^{\mathrm{T}}}{\partial K}\right\}\frac{\mathbf{MJ}}{\phi_q^{\mathrm{T}}\mathbf{M}\phi_q} + 2\phi_{iq}\frac{\phi_q^{\mathrm{T}}\mathbf{MJ}}{\{\phi_q^{\mathrm{T}}\mathbf{M}\phi_q\}^2}\left\{\frac{\partial \phi_q^{\mathrm{T}}}{\partial K}\mathbf{M}\phi_q\right\} \tag{35}$$

Equations (32)–(35) depend on the derivatives of squared modal frequencies, $\partial\omega_q^2/\partial K$, and mode shapes, $\partial\phi_q^{\mathrm{T}}/\partial K$, with respect to stiffness K. The reader is referred to Appendix III where expressions are given for the computation of such derivatives. Based on equations (31)–(35), the partial derivatives of the covariances between Fourier coefficients given in equations (16) and (17) can be evaluated and then the covariance matrix derivatives $\partial\mathbf{C}_k/\partial K$ can be assembled.

The stiffness K was assumed to have a prior lognormal probability density function. A prior mean $\mu_k = 3000$ K/in was considered for the stiffness. Two values were used for the prior coefficient of variation of the stiffness, COV $= 0{\cdot}15$ and $0{\cdot}30$. The corresponding prior lognormal distributions for K have parameters $\mu_{\ln K} = 7{\cdot}99$, $\sigma_{\ln K} = 0{\cdot}15$ and $\mu_{\ln K} = 7{\cdot}96$, $\sigma_{\ln K} = 0{\cdot}29$, respectively. The stiffness constants for the upper storeys c_i were taken all equal to 1·0 and therefore the distribution of mean stiffness with height is uniform. Records were assumed to have a sampling frequency of 50 Hz and a 10 sec duration.

Consider now the case where only one accelerometer is available and we want to choose its optimum location to best identify the stiffness K. The computation of the loss function expected value was carried out by means of Monte-Carlo simulations based on 3000 sample sets for K and K_i, $i = 2, \ldots, 5$. For each sample set and each possible location of the accelerometer, we proceed as follows: (1) solve the eigenvalue problem (equation (68) in Appendix III); (2) evaluate the modal transfer functions given in equation (25); (3) assemble the covariance matrices in equation (22) for the recorded response using equations (16) and (17), and computer their inverse; (4) evaluate the partial derivatives of squared modal frequencies and mode shape vectors solving the system of linear equations (71) in Appendix III; (5) evaluate the partial derivatives of modal transfer functions, modal coefficients and participation factors using equations (32)–(35); (6) assemble the covariance matrix partial derivatives; and (7) compute the Fisher information according to equation (23);

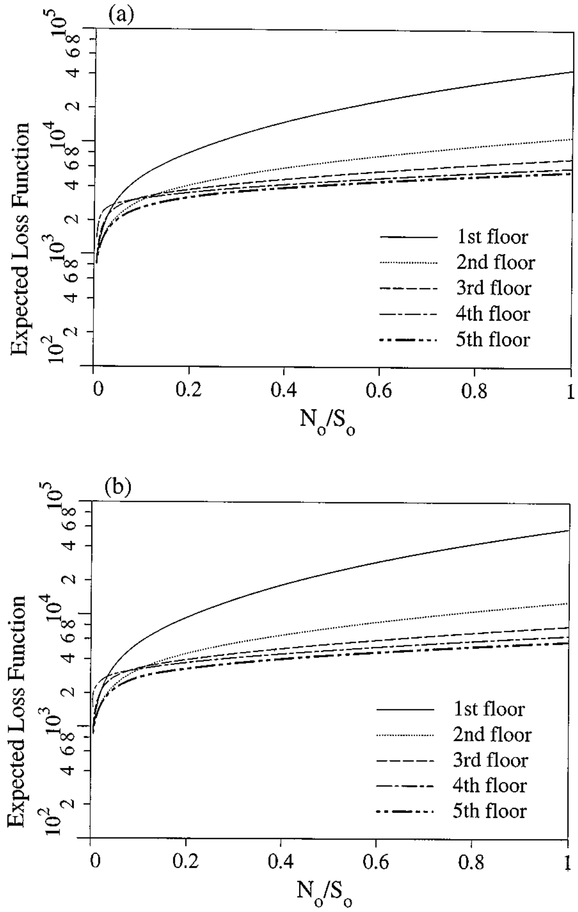

Figure 2. Expected loss function versus normalized noise: (a) COV = 0·15; (b) COV = 0·30

in this example, the Fisher information is a scalar given that there is only one uncertain structural parameter. Once steps 1–7 have been repeated for all the stiffness sample sets, the inverse of the Fisher information values are averaged to obtain the expected loss function.

Figure 2 shows the variation of the expected loss function, for each possible location of the accelerometer, versus a nomalized noise amplitude N_0/S_0, where N_0 and S_0 are the white noise amplitudes of the recording noise and of the seismic motion at baserock level, respectively. The results shown in Figure 2(a) were obtained for a stiffness prior coefficient of variation COV = 0·15. For all values considered of the normalized noise, the expected loss function has a minimum value when the accelerometer is placed at the fifth floor, although there is no significant difference from that corresponding to the fourth one. If the location of the accelerometer is moved downwards from the top, the expected loss function values become larger; hence, when comparing any two floors, a better location for the accelerometer is that of the floor that is atop. The expected loss function decreases with decreasing values of the normalized noise, i.e. for small expected recording noise amplitudes relative to those of the ground motion or, *vice versa*, for large expected ground motion amplitudes compared to those of the recording noise, as determined by the values of N_0 and S_0. As the normalized noise amplitude decreases towards zero, the expected loss functions for the recorded responses approach similar values; in the case of noise-free recordings, the accelerometer could be placed at

any of the floor levels. Figure 2(b) shows results for the case of a stiffness prior coefficient of variation COV $= 0{\cdot}30$. Results are similar to those shown in Figure 2(a) for COV $= 0{\cdot}15$. The COV in the prior distribution of K did not have an effect on the optimal location of the accelerometer; however, as discussed below, it did influence the reduction of prior uncertainty about K when the response is recorded.

The results shown in Figure 2 may be used to assess the amount of prior uncertainty about K that can be reduced by recording a response acceleration. Let $\mathrm{Var}(K\,|\,Y = y)$ denote the posterior variance of K when the values y are measured for response Y. Prior to the observation of Y, $\mathrm{Var}(K\,|\,Y)$ is an uncertain variable. The posterior variance of K, averaged over all possible values of the response Y is then $E[\mathrm{Var}(K\,|\,Y)]$. It is worth pointing out that $E[\mathrm{Var}(K\,|\,Y)]$ is the posterior variance associated with the process of recording Y and then estimating K, *before* any particular value $Y = y$ has been observed. Thus, if response Y is recorded for estimating K, the reduction in the uncertainty about K is then $\mathrm{Var}(K) - E[\mathrm{Var}(K\,|\,Y)]$, where $\mathrm{Var}(K)$ is the prior variance of K. Given that the Bayesian loss function used here is the squared error function, $L(K, \hat{K}) = \{K - \hat{K}(Y)\}^2$, the estimator $\hat{K}(Y)$ is equal to the conditional mean of K given the response Y, i.e. $\hat{K}(Y) = E[K\,|\,Y]$. Hence,

$$\begin{aligned} E[\mathrm{Var}(K\,|\,Y)] &= E_Y E_{K|Y}[\{K - \hat{K}(Y)\}^2] \\ &= E[L(K, \hat{K})] \\ &= E_K M_K^{-1} \end{aligned} \tag{36}$$

where M_K is the Fisher information in Y about K. As seen in equation (36), the Fisher information may be used to evaluate the posterior variance of K and, therefore, to assess the reduction of prior uncertainty associated with the process of recording response Y for the estimation of K. In fact, if an instrumentation program is to be useful, a reduction of prior uncertainty about the structural parameters should be achieved based on the information contained in the recordings.

Figure 2 thus shows that the posterior stiffness variance increases with increasing values of the normalized noise; the larger the noise amplitudes the lower the reduction of prior uncertainty. For relatively large recording noise amplitudes or for low to moderate ground motion intensities, the most accurate estimates of the stiffness can be expected if the accelerometer is placed at the fifth floor to record the response. On the other hand, Figure 2 also shows that for small values of the normalized noise, say $N_0/S_0 \leqslant 0{\cdot}05$, i.e. for noise-free recordings or large ground motion intensities, there is not a significant difference in the posterior stiffness variances, thus implying, as stated before, that the accelerometer could be placed at any of the five floors. However, taking into account that: (1) the ground motion white noise intensity S_0 of future earthquakes is uncertain, and (2) once the accelerometer has been placed on the structure one would expect the recordings obtained during any future event (even if it corresponds to a low to moderate intensity S_0) to be useful for reducing as much of the prior uncertainty as possible, then the optimal location for the accelerometer is that of the fifth floor.

One should keep in mind that these results depend not only on the expected ground motion amplitudes, as measured by S_0, but also on its dominant or characteristic frequency. Figure 3 shows results for the same example structure but subjected to a random earthquake ground motion excitation with characteristic soil frequency $\omega_g = 2{\cdot}47$ Hz and critical damping ratio $\xi_g = 0{\cdot}20$ in the Kanai-Tajimi spectral density function. The characteristic frequency ω_g was chosen to match the mean first modal frequency of the structural system so that the contribution of the first mode to the total response was dominant. The results in Figure 3 show that when the structure tends to respond in its first fundamental mode, there will not be much of a difference in the expected loss function values associated with the possible locations of the accelerometer. Hence, it should be just as useful in this case to place the accelerometer at any of the floor levels. These results suggest that when excited in the fundamental mode, response amplitudes are much larger than those of noise. It is also interesting to point out that a lower posterior uncertainty about K can be expected when records are obtained from the structure responding in its fundamental mode. In the following, the characteristic frequency in the Kanai-Tajimi spectral density is kept equal to $0{\cdot}8$ Hz as assumed first.

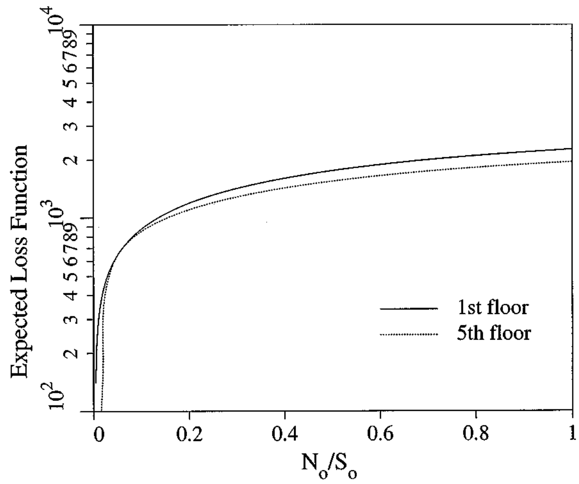

Figure 3. Expected loss function versus normalized noise, $\omega_g = 2{\cdot}47$ Hz, COV = 0·15

Consider now an uncertainty reduction index (URI) defined as the portion of the prior coefficient of variation that can be reduced by means of an instrumentation program,

$$\mathrm{URI} = \frac{\mathrm{COV} - \mathrm{COV_f}}{\mathrm{COV}} \tag{37}$$

where $\mathrm{COV_f}$ is the posterior coefficient of variation of the stiffness K. The larger the URI, the more efficient the instrumentation is regarding the reduction of prior uncertainty about K. Optimally, the URI should be as close to one as possible. Figure 4(a) shows plots of the URI versus the normalized noise amplitudes, when the accelerometer is placed at the first floor and for both prior coefficients of variation COV = 0·15 and 0·30. It can be seen that a greater uncertainty reduction can be achieved when the prior uncertainty is higher. Similar results are shown in Figure 4(b) when the accelerometer is placed at the fifth floor. It is interesting to notice that a very small posterior uncertainty about K remains once recordings from the fifth floor are obtained for the estimation of K. Hence, one can expect to be able to estimate, with great certainty, the lateral stiffness from records of the structure considered in this example. Different results in terms of optimal accelerometer location and uncertainty reduction might be obtained for the case of partially correlated lateral stiffnesses or for the case where other structural parameters, such as masses and dampings, were to be estimated from a response recording. Moreover, other locations could be found to be optimal if considerations of uniqueness in the solution of the inverse problem for identification of the lateral stiffness were taken into account.[3,4]

The criterion proposed in this paper can be used to assess the influence of the duration of the recordings on the reduction of prior uncertainty. Following the same procedure indicated before for the computation of the expected loss function, results were obtained for different recording durations. Figure 5 shows the variation of the URI as a function of the record durations for a normalized noise amplitude of 0·20 and for prior coefficients of variation COV = 0·15 and 0·30. As expected, the URI increases with longer record durations. Take, for instance, the case where the accelerometer is placed at the fifth floor and COV = 0·15 (Figure 5(a)): the URI increases from values of about 60 per cent for a record duration equal to 2 sec, up to values of about 85 per cent for record durations equal to 10 sec. These curves indicate the minimum duration that records should have in order to obtain a significant reduction of prior uncertainty. For instance, for a prior COV = 0·15 and a 2 sec record duration, the URI would be equal to about 25 per cent if the accelerometer had been placed at the first floor. Given the noise amplitude considered, records with shorter durations would not be useful for identifying the lateral stiffness. Between 2 and 10 sec, the increase in the URI is

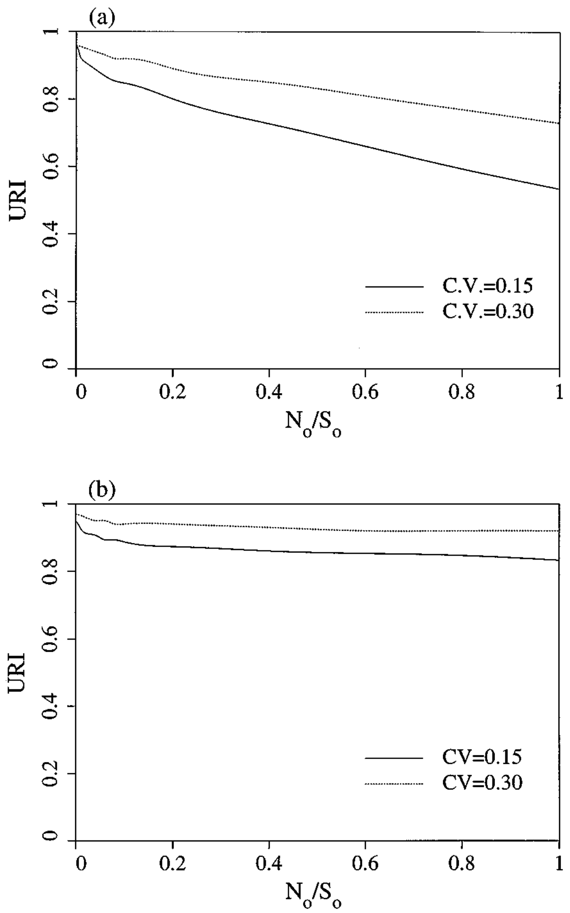

Figure 4. Uncertainty reduction index versus normalized noise for accelerometer located at (a) first floor; (b) fifth floor

significant. For a target uncertainty reduction, say 80 per cent, acceleration records from the first floor should have a duration of at least 10 sec.

The capabilities of the criterion were also tested against cases in which the distribution of mean lateral stiffness with height is not uniform. Four cases with different sets of values for the stiffness constants c_i were used to obtain structures with different configurations of mean stiffness with height: Case I: $c_2 = 100, c_3 = 1, c_4 = 1, c_5 = 100$; Case II: $c_2 = 100, c_3 = 100, c_4 = 1, c_5 = 1$; Case III: $c_2 = 100, c_3 = 100, c_4 = 100, c_5 = 100$; Case IV: $c_2 = 1$, $c_3 = 0{\cdot}01$, $c_4 = 1$, $c_5 = 1$. The expected loss functions were computed assuming a prior COV $= 0{\cdot}15$, a normalized noise $N_0/S_0 = 0{\cdot}20$ and a record duration equal to 10 sec. Results are listed in Table I and Figure 6 shows the optimal location of the accelerometer for each of the four cases. In Case I ($K_2 = 100K, K_3 = K, K_4 = K, K_5 = 100K$), the second and fifth inter-storeys are stiff compared to the other ones. Due to its relative large stiffness, response accelerations along the degrees of freedom at the fourth and fifth floors are highly correlated and the results show that the expected loss function values when the acelerometer is placed at any of these floors are consequently very close. The same argument can be stated for the response acceleration at the first and second floors. The results show that either the fourth or the fifth floor is optimal location for the accelerometer, implying that their responses are more sensitive to the lateral stiffness than those of the first and second floors. In Case II ($K_2 = 100K$, $K_3 = 100K$, $K_4 = K$, $K_5 = K$),

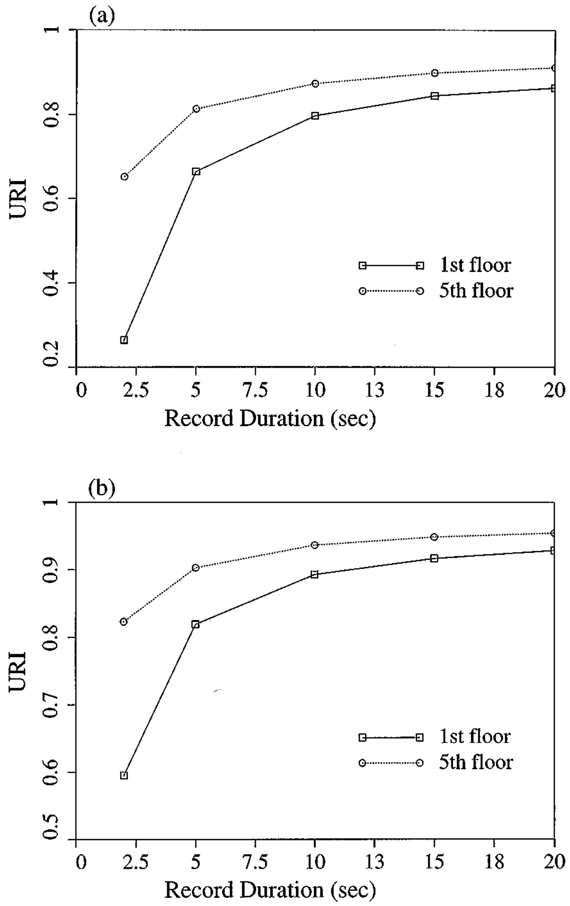

Figure 5. Uncertainty reduction index versus record duration: (a) COV = 0·15; COV = 0·30

Table I. Expected loss functions for a structure with variation of mean stiffness with height

Storey level	Case I	Case II	Case III	Case IV
1st	5172	4400	3411	6001
2nd	5122	4362	3388	3079
3rd	3629	4335	3371	14 066
4th	3071	3155	3359	14 081
5th	3068	2720	3354	14 087

response accelerations along the degrees of freedom at the first, second and third floor levels are highly correlated; hence, the computed expected loss functions when the accelerometer is placed at any of these floors are similar to each other. As in the case of uniform mean stiffness with height, the optimal location for the accelerometer is the fifth floor. In Case III ($K_2 = 100K$, $K_3 = 100K$, $K_4 = 100K$, $K_5 = 100K$), the four upper storeys tend to respond as a rigid block. Although the expected loss function when the accelerometer is

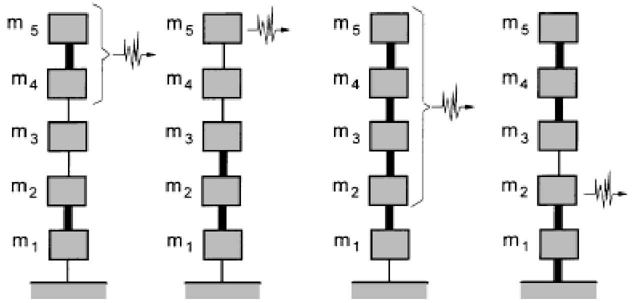

Figure 6. Optimal location of accelerometer for different distributions of mean stiffness with height; COV = 0·15

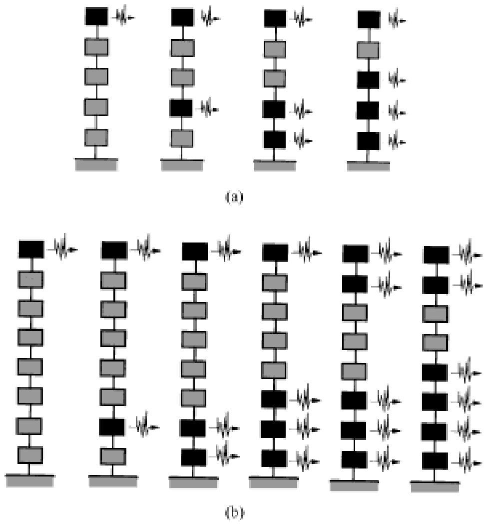

Figure 7. Optimal arrays of accelerometers: (a) 5-DOF system; (b) 8-DOF system

placed at the fifth floor is the minimum, as a consequence of its being more sensitive to K, it is seen that values of the loss function at the other floor levels are close to that corresponding to the fifth one. Case IV yields a different optimal location for the accelerometer. In Case IV ($K_2 = K$, $K_3 = 0.01K$, $K_4 = K$, $K_5 = K$), the third inter-storey is very flexible compared to the stiffness of the other inter-storeys. Due to such flexibility, a small portion of the ground motion energy input excites the responses of the upper three storeys, which are highly correlated as indicated by the values of the corresponding expected loss functions. The response acceleration along the second and first floor degrees of freedom are less correlated compared to those of the upper storeys. The minimum expected loss function corresponds to an accelerometer placed at the second floor which becomes the optimal location and suggests that such response is more sensitive to changes in K than that of the first floor.

In the case of the example structure with known masses and damping, and equal mean inter-storey stiffnesses, that has been discussed here, the prior uncertainty about K can be reduced significantly if just one accelerometer is placed on the structure. However, just for the sake of completeness and illustration of the criterion capabilities, results are given now for the cases in which more than one accelerometer was available. Figure 7(a) shows the optimal location for different numbers of accelerometers considering a normalized noise equal to 0·20, a 10 sec record duration and COV = 0·15. It is seen that if one accelerometer were added

at a time to the array of instruments, it should be placed on the structure in the following order: first at the second floor, then at the first one and finally at the third one. The case of a similar eight-storey shear building subjected to the same ground motion excitation was also analysed. Masses concentrated at the floor levels were taken all equal to 1 K $\sec^2$/in, critical damping ratios of 2 per cent were given for all modes, a prior stiffness COV = 0·15 was considered, and the distribution of mean lateral stiffness with height was assumed uniform, $c_i = 1$, $i = 2, \ldots, 8$. A normalized noise amplitude equal to 0·20 and a 10 sec record duration were also considered. The optimal locations for different number of accelerometers are shown in Figure 7(b). As in the case of the five-degree-of-freedom building, the top floor is the optimal location for a single accelerometer. Next optimal locations are the second, first and third floors. From then on, the results suggest that further accelerometers should be placed starting from the top and lower storeys moving downwards and upwards, respectively. Similar results have also been obtained for systems with a greater number of degrees of freedom.

CONCLUSIONS

A criterion has been proposed for selecting the optimal location of a given number of sensors to record the seismic response of a structural system for identification purposes. The optimal sensor locations are selected so that the expected value of a Bayesian loss function is minimized. Assuming that efficient estimators are used for the structural parameters, the expected loss function is then given in terms of the expected size of the Fisher information matrix associated with the responses to be recorded. The criterion has been applied to the case of linear systems with uncertain structural properties subjected to random earthquake ground motions. By means of a discrete Fourier transform, responses were expressed in terms of their Fourier coefficients and the problem formulated in the frequency domain. Thus, an efficient use can be made of the statistical independence between response Fourier coefficients and of the stochastic solution for the structural response in terms of cross-spectral density functions. Expressions have been derived for computing the Fisher information matrix associated with a set of recorded responses. The formulation clearly shows that the Fisher information matrix is a weighted combination of the covariances between recorded responses and the sensitivity of such covariances to changes in the uncertain structural parameters. Thus, the relative weights of the covariances between responses and of the sensitivities of such covariances to the structural parameters to be identified, can be accounted for in the selection of the optimal locations of the recording instruments, which is a fundamental contribution of the paper.

A thorough example has been given to illustrate the use and features of the proposed formulation. Results in terms of optimal locations of recording instruments and reduction of uncertainty about the structural properties, have been used to assess the influence of the recording noise, the ground-motion-dominant frequencies and expected amplitudes, the record durations, the prior uncertainty in the knowledge of the structural properties and the correlation structure of the recorded responses. In the case of shear buildings where the uncertain lateral stiffnesses may be considered to be perfectly correlated, the optimal location for a single accelerometer is that of the top storey. Much of the prior uncertainty is found to be reduced by recordings from a single accelerometer. However, different conclusions may be drawn in those cases where other structural properties are to be identified instead, or when the assumption of perfect correlation could not hold. In its present form, the formulation proposed here can be used to analyse such cases which are left as the topic of future studies by the authors. Optimal arrays for a number of accelerometers have also been proposed. The results suggest that if several accelerometers are available, they should be placed starting at the top and bottom floors, and then progress with the instrumentation towards the middle ones.

APPENDIX I

Here we will show that

$$E_{Y|\Theta}\left[\mathbf{Y}_i^{\mathrm{T}}\frac{\partial \mathbf{C}_i^{-1}}{\partial \theta_k}\mathbf{Y}_i\right] = -\frac{\partial \ln \Delta_i}{\partial \theta_k} \tag{38}$$

For the simplicity of notation in the proof that follows, the subscript 'i' will be dropped from the terms in equation (38). Thus, let $\mathbf{C}$ denote the conditional covariance matrix of a random vector $\mathbf{Y}$, $\mathbf{C} = E_{Y|\Theta}[\mathbf{Y}\mathbf{Y}^{\mathrm{T}}]$, and let Δ denote its determinant. By the chain rule of differentiation, and using indicial notation, we have that

$$\frac{\partial \ln \Delta}{\partial \theta_k} = \frac{\partial \ln \Delta}{\partial C_{ij}} \frac{\partial C_{ij}}{\partial \theta_k} \tag{39}$$

where C_{ij} is the element in the ith row and jth column of the covariance matrix $\mathbf{C}$. If matrix $\bar{\mathbf{C}}$ denotes the adjoint of $\mathbf{C}$, the inverse matrix $\mathbf{C}^{-1}$ is given by

$$\mathbf{C}^{-1} = \frac{\bar{\mathbf{C}}}{\Delta} \tag{40}$$

Since the product $\mathbf{C}\mathbf{C}^{-1}$ equals the identity matrix $\mathbf{I}$, then

$$\Delta \mathbf{I} = \mathbf{C}\bar{\mathbf{C}} \tag{41}$$

Considering the ith element in the diagonal of matrix $\Delta\mathbf{I}$, and using incidal notation, it follows from equation (41) that the determinant Δ can be expressed as

$$\Delta = C_{ik}\bar{C}_{ki} \tag{42}$$

where $\bar{C}_{ki}$ is an element in matrix $\bar{\mathbf{C}}$. Equation (42) can be used to obtain the partial derivative

$$\frac{\partial \ln \Delta}{\partial C_{ij}} = \frac{1}{\Delta} \left\{ \frac{\partial C_{ik}}{\partial C_{ij}} \bar{C}_{ki} + C_{ik} \frac{\partial \bar{C}_{ki}}{\partial C_{ij}} \right\} \tag{43}$$

Taking into account that (1) $\partial C_{ik}/\partial C_{ij}$ equals the Kronecker δ_{jk}, and (2) $\partial \bar{C}_{ki}/\partial C_{ij} = 0$ since by definition of adjoint matrix, element $\bar{C}_{ki}$ does not depend on the elements in the ith row of $\mathbf{C}$, then equation (43) reduces to

$$\frac{\partial \ln \Delta}{\partial C_{ij}} = \frac{1}{\Delta} \bar{C}_{ji} \tag{44}$$

Noting that, by definition of inverse matrix, the right-hand side of equation (44) is the jith element of matrix $\mathbf{C}^{-1}$ and given the symmetry of covariance matrix $\mathbf{C}$, then

$$\frac{\partial \ln \Delta}{\partial C_{ij}} = C_{ij}^{-1} \tag{45}$$

and substituting equation (45) back into equation (39),

$$\frac{\partial \ln \Delta}{\partial \theta_k} = C_{ij}^{-1} \frac{\partial C_{ij}}{\partial \theta_k} \tag{46}$$

According to indicial notation, the product in equation (46) represents a dot product between matrices $\mathbf{C}^{-1}$ and $\partial \mathbf{C}/\partial \theta_k$,

$$\frac{\partial \ln \Delta}{\partial \theta_k} = \mathbf{C}^{-1} \cdot \partial \mathbf{C}/\partial \theta_k$$

Consider now the expected value in equation (38) expressed in indicial notation,

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}} \frac{\partial \mathbf{C}^{-1}}{\partial \theta_k} \mathbf{Y} \right] = E_{Y|\Theta}\left[Y_m \frac{\partial C_{mn}^{-1}}{\partial \theta_k} Y_n \right] \tag{47}$$

As stated above, the subscript has been dropped for simplicity on the left-hand side of equation (47). Given that the expectation is taken conditionally on Θ, equation (47) can be written as follows:

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\right] = \frac{\partial C_{mn}^{-1}}{\partial \theta_k} E_{Y|\Theta}[Y_m Y_n]$$
$$= \frac{\partial C_{mn}^{-1}}{\partial \theta_k} C_{mn} \tag{48}$$

where the last equality in equation (48) follows from the definition of covariance, $E_{Y|\Theta}[Y_m Y_n] = C_{mn}$. From the chain rule for the derivative of a product, the right-hand side in equation (48) can be expressed as

$$\frac{\partial C_{mn}^{-1}}{\partial \theta_k} C_{mn} = \frac{\partial}{\partial \theta_k}\{C_{mn}^{-1} C_{mn}\} - C_{mn}^{-1}\frac{\partial C_{mn}}{\partial \theta_k} \tag{49}$$

The first partial derivative on the right-hand side of equation (49) is equal to zero since the term $C_{mn}^{-1} C_{mn}$ represents the trace of the identity matrix $\mathbf{I}$,

$$C_{mn}^{-1} C_{mn} = C_{mn}^{-1} C_{nm}$$
$$= I_{mm}$$
$$= \operatorname{tr} \mathbf{I} \tag{50}$$

Thus,

$$\frac{\partial C_{mn}^{-1}}{\partial \theta_k} C_{mn} = -\, C_{mn}^{-1}\frac{\partial C_{mn}}{\partial \theta_k} \tag{51}$$

Substituting equation (51) into equation (48) one obtains

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\right] = -\, C_{mn}^{-1}\frac{\partial C_{mn}}{\partial \theta_k} \tag{52}$$

and from equation (46) it follows that

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\right] = -\frac{\partial \ln \Delta}{\partial \theta_k} \tag{53}$$

which proves expression (10) used in this paper.

APPENDIX II

Here we prove the following expression used in equation (12):

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\,\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_l}\mathbf{Y}\right] = 2\mathbf{C}^{-1}\frac{\partial \mathbf{C}}{\partial \theta_k}\cdot\frac{\partial \mathbf{C}}{\partial \theta_l}\mathbf{C}^{-1} + \frac{\partial \ln \Delta}{\partial \theta_k}\frac{\partial \ln \Delta}{\partial \theta_l} \tag{54}$$

In equation (54), $\mathbf{C}$, $\mathbf{Y}$ and Δ are defined as in Appendix I. Using indicial notation and denoting by K_{qm} and L_{np} the elements in matrices $\partial \mathbf{C}^{-1}/\partial \theta_k$ and $\partial \mathbf{C}^{-1}/\partial \theta_l$, respectively, equation (54) can be expressed as

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\,\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_l}\mathbf{Y}\right] = E_{Y|\Theta}[Y_q K_{qm} Y_m Y_n L_{np} Y_p] \tag{55}$$

The expected value in equation (55) involves fourth-order moments that can be obtained based on second-order moments provided $\mathbf{Y}$ is a Gaussian random vector as follows:

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\,\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_l}\mathbf{Y}\right] = K_{qm}L_{np}E_{Y|\Theta}[Y_q Y_m Y_n Y_p]$$
$$= K_{qm}L_{np}\{C_{qm}C_{np} + C_{qn}C_{mp} + C_{qp}C_{mn}\} \qquad (56)$$

where $C_{ij} = E_{Y|\Theta}[Y_i Y_j]$ is an element of covariance matrix $\mathbf{C} = E_{Y|\Theta}[\mathbf{Y}\mathbf{Y}^{\mathrm{T}}]$. Since $C_{ij} = C_{ji}$ and $L_{np} = L_{pn}$, because of symmetry, equation (56) is then equal to

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\,\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_l}\mathbf{Y}\right] = K_{qm}C_{qm}L_{np}C_{np} + K_{qm}C_{qn}L_{np}C_{pm} + K_{qm}C_{qp}L_{pn}C_{nm} \qquad (57)$$

Notice that the matrix products $C_{qn}L_{np}C_{pm}$ and $C_{qp}L_{pn}C_{nm}$ are equal to each other; thus equation (57) reduces to

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\,\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_l}\mathbf{Y}\right] = K_{qm}C_{qm}L_{np}C_{np} + 2K_{qm}C_{qn}L_{np}C_{pm} \qquad (58)$$

Recalling that K_{qm} and L_{np} denote elements of matrices $\partial \mathbf{C}^{-1}/\partial\theta_k$, $\partial \mathbf{C}^{-1}/\partial\theta_l$, respectively, the first term on the right-hand side of equation (58) is equal to

$$K_{qm}C_{qm}L_{np}C_{np} = \frac{\partial C_{qm}^{-1}}{\partial \theta_k}C_{qm}\frac{\partial C_{np}^{-1}}{\partial \theta_l}C_{np} \qquad (59)$$

It follows from equations (46) and (51) in Appendix I that

$$\frac{\partial \ln \Delta}{\partial \theta_k} = C_{ij}^{-1}\frac{\partial C_{ij}}{\partial \theta_k} = -\frac{\partial C_{ij}^{-1}}{\partial \theta_k}C_{ij} \qquad (60)$$

Thus, equation (59) becomes

$$K_{qm}C_{qm}L_{np}C_{np} = \frac{\partial \ln \Delta}{\partial \theta_k}\frac{\partial \ln \Delta}{\partial \theta_l} \qquad (61)$$

Consider now the term $L_{np}C_{pm}$ on the right-hand side of equation (58); it can be written as

$$L_{np}C_{pm} = \frac{\partial C_{np}^{-1}}{\partial \theta_l}C_{pm} \qquad (62)$$

and according to the derivative of a product,

$$L_{np}C_{pm} = \frac{\partial}{\partial \theta_l}\{C_{np}^{-1}C_{pm}\} - C_{np}^{-1}\frac{\partial C_{pm}}{\partial \theta_l}$$
$$= -C_{np}^{-1}\frac{\partial C_{pm}}{\partial \theta_l} \qquad (63)$$

given that $C_{np}^{-1}C_{pm} = \delta_{nm}$, δ_{nm} being the Kronecker delta. In the same way, it can be shown that

$$K_{qm}C_{qn} = -C_{mq}^{-1}\frac{\partial C_{qn}}{\partial \theta_k} \qquad (64)$$

Using equations (63) and (64), the term $K_{qm}C_{qn}L_{np}C_{pm}$ on the right-hand side of equation (58) is given by

$$K_{qm}C_{qn}L_{np}C_{pm} = C_{mq}^{-1}\frac{\partial C_{qn}}{\partial \theta_k}C_{np}^{-1}\frac{\partial C_{pm}}{\partial \theta_l} \qquad (65)$$

The expression in indicial notation in equation (65) represents the following matrix product:

$$K_{qm}C_{qn}L_{np}C_{pm} = \mathbf{C}^{-1}\frac{\partial \mathbf{C}}{\partial \theta_k}\cdot\left\{\mathbf{C}^{-1}\frac{\partial \mathbf{C}}{\partial \theta_l}\right\}^{\mathrm{T}}$$

$$= \mathbf{C}^{-1}\frac{\partial \mathbf{C}}{\partial \theta_k}\cdot\frac{\partial \mathbf{C}}{\partial \theta_l}\mathbf{C}^{-1} \tag{66}$$

Therefore, substituting equations (61) and (66) into equation (58), one obtains

$$E_{Y|\Theta}\left[\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_k}\mathbf{Y}\,\mathbf{Y}^{\mathrm{T}}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_l}\mathbf{Y}\right] = 2\mathbf{C}^{-1}\frac{\partial \mathbf{C}}{\partial \theta_k}\cdot\frac{\partial \mathbf{C}}{\partial \theta_l}\mathbf{C}^{-1} + \frac{\partial \ln \Delta}{\partial \theta_k}\frac{\partial \ln \Delta}{\partial \theta_l} \tag{67}$$

which is the expression used in equation (12) in this paper.

APPENDIX III

Modal frequencies and mode shapes are obtained from the eigenvalue solution of the system of equations,

$$\{\mathbf{K} - \omega^2\mathbf{M}\}\boldsymbol{\phi} = \mathbf{0} \tag{68}$$

Let K_{ij}, M_{ij} denote elements of the stiffness and mass matrices $\mathbf{K}$, $\mathbf{M}$, respectively, and ϕ_j denote a component of mode shape vector $\boldsymbol{\phi}$. Using indicial notation the ith equation in equation (68) can be expressed as

$$K_{ij}\phi_j - \omega^2 M_{ij}\phi_j = 0 \tag{69}$$

Taking partial derivatives in equation (69) one obtains

$$K_{ij}\phi_{j,K} - \omega^2 M_{ij}\phi_{j,K} - \omega^2_{,K}M_{ij}\phi_j = -K_{ij,K}\phi_j \tag{70}$$

where the subscript ',K' denotes partial derivative of the variable subscripted with respect to K. Noticing that (1) for a system with concentrated masses $\omega^2_{,K}M_{ij}\phi_j = \omega^2_{,K}m_j\phi_j$, where m_j is the jth element in the diagonal of matrix $\mathbf{M}$; and (2) the first mode shape vector component is usually normalized to one as auxiliary condition to solve the eigenvalue problem in equation (68), i.e. $\phi_{1,K} = 0$, the system of equations (70) can be written in matrix form as follows:

$$\begin{bmatrix} -m_1\phi_1 & K_{12} & K_{13} & \cdots & K_{1Q} \\ -m_2\phi_2 & K_{22}-\omega^2 m_2 & K_{23} & \cdots & K_{2Q} \\ -m_3\phi_3 & K_{32} & K_{33}-\omega^2 m_3 & \cdots & K_{3Q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_Q\phi_Q & K_{Q2} & K_{Q3} & \cdots & K_{QQ}-\omega^2 m_Q \end{bmatrix}\begin{bmatrix} \omega^2_{,K} \\ \phi_{2,K} \\ \phi_{3,K} \\ \vdots \\ \phi_{Q,K} \end{bmatrix} = -\frac{\partial \mathbf{K}}{\partial K}\boldsymbol{\phi} \tag{71}$$

Solving the system of linear equations (71) yields the partial derivatives of modal frequencies and mode shape vectors with respect to the first storey lateral stiffness K.

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